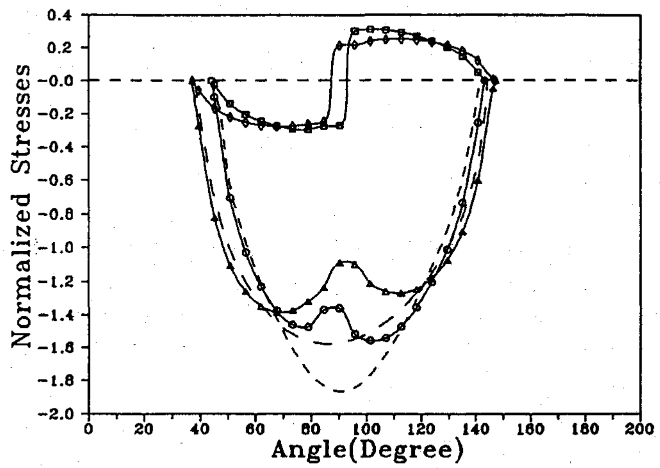

Fig. 3 Contact stress distribution of $[0_2/\pm 45\ \text{deg}]_s$ laminate for variation of E in the case of two pins in parallel: case I) E/R = 6; and case II) E/R = 3: - - - -, nonfriction (case I); -o-o-o-, friction normal (case I); -□-□-□-, tangential (case I); — — —, nonfriction (case II); -△-△-△-, friction normal (case II); and -◇-◇-◇-, tangential (case II).

parallel is discretized by 546 four-node linear elements with 613 nodes by utilizing symmetry. In Fig. 3, the results for two values of distance between the upper edge of the plate to the center of hole E are shown for the $[0_2/\pm 45\ \text{deg}]_s$ laminate. As E becomes smaller, maximum normal contact stress is reduced and the point of sign change of tangential stress is located outward from the symmetric axis of the plate. Also, a wider contact area is found as E becomes smaller. Also, our computations show the following effects of distance from the symmetric axis of the plate to the hole G: both contact areas are almost the same, but the maximum normal contact stress moves toward the symmetric axis of the plate as G becomes smaller.

The results indicate the importance of geometric factors in the design of pin-joint structures. In all cases, $P = 8000$ N, $\lambda = 0.1\%$, and $\mu = 0.2$ are assumed.

Conclusions

This study investigated the parametric behavior of multipin joints of laminated composite plates. Clearances, friction coefficients, and geometric factors are considered as design parameters for the analysis. Using the penalty finite element method, the pin-joint problems are solved.

Through extensive parametric study, the following can be concluded: Friction and clearance have a significant influence on the distribution and the maximum value of the contact stresses. Friction changes the location of the maximum contact stresses and increases contact area. Clearance affects contact area and the maximum contact stress. Note that geometric factors also significantly influence the distribution of contact stresses. These factors change the location of maximum contact stress and contact area. Also, pin loadings are changed significantly with variation of the above mentioned parameters in case of two pins in series. These findings will be useful for the design of composite structures by the aerospace industry.

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New Method for Deriving Eigenvalue Rate with Respect to Support Location

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I. Introduction

EIGENVALUE rate with respect to cross-sectional variables and shape variables is of importance in design of structural vibrational systems and, thus, attracts much attention in the research fields.[1–3] Though eigenvalue rate with respect to the cross-sectional variables has been widely and deeply investigated, eigenvalue rate with respect to the shape variables is less evolved. Very recently, Wang[4] used the classical normal modal method to derive the formulas of eigenvalue rate with respect to in-span support; Hou and Chuang[5] used the material derivative method to derive the formulas of eigenvalue rate with respect to in-span support. Their results with the beam as an example show that the eigenvalue rate with respect to in-span support location is proportional to the slope of the corresponding mode shape and the support reaction force. In this Note, the authors present a new method to derive the formulas of eigenvalue rate with respect to in-span support location using the generalized variational principle of the Rayleigh quotient. Following the ideas of this Note, the eigenvalue rate, with respect to such variables as location of discrete or continuous in-span supports, the location of concentrated mass, location of elastic support, and location of a substructure, can be derived without any conceptual difficulties. At the end of this Note, a numerical comparison is given to illustrate the application of the method presented.

II. Problem Definition

For the sake of simplicity, let us take a beam, shown in Fig. 1, as an example to illustrate the new method for deriving the formulas of eigenvalue rate with respect to in-span support location. The beam

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shown in Fig. 1 has an in-span rigid support with parameter b as its location coordinate. The Rayleigh quotient for the beam can be expressed as

$$R[y(x)] = \Pi[y(x)]/T[y(x)] \tag{1}$$

where the functional $R[y(x)]$ is referred to as the Rayleigh quotient, functional $\Pi[y(x)]/2$ and $T[y(x)]/2$ are the potential energy and the coefficient of kinetic energy of the beam, respectively. $\Pi[y(x)]$ and $T[y(x)]$ are given by

$$\Pi[y(x)] = \int_0^l EJ[y''(x)]^2\,dx \tag{2}$$

$$T[y(x)] = \int_0^l my^2(x)\,dx \tag{3}$$

The variational principle of the Rayleigh quotient says that the stationary value of $R[y(x)]$ gives the eigenvalue (square of natural frequency),

$$\omega^2 = st\,R[y(x)] = st\,\Pi[y(x)]/T[y(x)] \tag{4}$$

where the unknown $y(x)$ is subject to an additional constraint

$$y(b) = 0 \tag{5}$$

Denoting $\hat{y}(x)$ as the eigenfunction of Eq. (4), we have

$$\omega^2 = R[\hat{y}(x)] = \Pi[\hat{y}(x)]/T[\hat{y}(x)] \tag{6}$$

As can be seen from Eqs. (4–6), ω^2 is related to the support location parameter b in terms of Eq. (5). Eigenvalue ω^2 is implicitly dependent on b, however, and hence cannot be expressed explicitly in terms of parameter b.

The problem to be studied in this paper can be described as: provided that the ω^2 and the associated eigenfunction $\hat{y}(x)$ have been obtained, how can one obtain $\delta\omega^2$, the first-order variation of ω^2, when the support location b receives a variation δb?

III. Solution Approach

The main idea of this Note is to cast $y(b) = 0$ into the functional expression of $R[y(x)]$ with $y(b) = 0$ removed from the additional constraint. This can be realized by means of the generalized variational principle.

We can introduce the constraint $y(b) = 0$ by forming another functional,

$$G[y(x), \lambda] = \frac{\Pi[y(x)] + 2\lambda y(b)}{T[y(x)]} \tag{7}$$

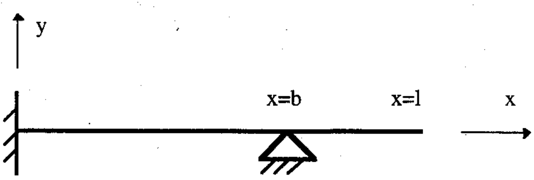

Fig. 1 Beam with an in-span support.

in which λ is an independent coordinate known as the Lagrange multiplier. The variation of the new functional is now

$$\delta G = \delta R + \frac{2y(b)\delta\lambda + 2\lambda\delta y(b)}{T[y(x)]} \tag{8}$$

and this is zero providing $y(b) = 0$ and, simultaneously, $\delta R = 0$.

It has been proven[1] that the two functional G and R have the same eigendata. Denoting $\hat{y}(x)$ and $\hat{\lambda}$ as the eigenfunction and the right Lagrange multiplier, respectively, that make $G[y(x), \lambda]$ stationary, we have

$$\omega^2 = G[\hat{y}(x), \hat{\lambda}] = \frac{\Pi[\hat{y}(x)] + 2\hat{\lambda}\hat{y}(b)}{T[\hat{y}(x)]} \tag{9}$$

Since the parameter b has been included in $G[\hat{y}(x), \hat{\lambda}]$, Eq. (9) can be differentiated with respect to b to obtain the eigenvalue rate with respect to b.

Suppose the support location b receives a variation δb, causing variation $\delta\hat{y}(x)$ and $\delta\hat{\lambda}$ of the eigenfunction and the right Lagrange multiplier, respectively. One can regard Eq. (9) as a functional with three arguments: $\hat{y}(x)$, $\hat{\lambda}$, and b, so that $\delta\omega^2$ can be expressed in three terms as

$$\delta\omega^2 = \delta_y\omega^2 + \delta_\lambda\omega^2 + \delta_b\omega^2 \tag{10}$$

where $\delta_b\omega^2$ denotes the variation of ω^2 induced by the variation b alone, keeping $\hat{y}(x)$ and $\hat{\lambda}$ unchanged, i.e.,

$$\delta_b\omega^2 = \frac{2\hat{\lambda}}{T[\hat{y}(x)]}\hat{y}(b)\delta b \tag{11}$$

$\delta_y\omega^2$ denotes the variation ω^2 induced by the variation $\hat{y}(x)$ alone, keeping $\hat{\lambda}$ and b unchanged; $\delta_\lambda\omega^2$ denotes the variation ω^2 induced by the variation $\hat{\lambda}$ alone, keeping $\hat{y}(x)$ and b unchanged.

Since $\hat{y}(x)$ and $\hat{\lambda}$ make the right side of Eq. (9) stationary, for any $\delta\hat{y}$ and any $\delta\hat{\lambda} \in R$, one has

$$\delta_y\omega^2 = \delta_\lambda\omega^2 = 0 \tag{12}$$

Therefore, one gets a simple formula,

$$\delta\omega^2 = \delta_b\omega^2 = \frac{2\hat{\lambda}}{T[\hat{y}(x)]}[\hat{y}'(b)]\delta b \tag{13}$$

The unknown Lagrange multiplier $\hat{\lambda}$ in the right side of Eq. (13) must be identified before Eq. (13) is used for computation purpose. Using the variational conditions of functional $G[y(x), \lambda]$, $\hat{\lambda}$ can be easily identified,[1]

$$\hat{\lambda} = -\left\{\frac{d}{dx}[EJ\hat{y}''(x)]|_{x=b+0} - \frac{d}{dx}[EJ\hat{y}''(x)]|_{x=b-0}\right\} \tag{14}$$

As can be seen from Eq. (14), $\hat{\lambda}$ is, in fact, the reaction force exerted by the support. Thus, $\hat{\lambda}$ can also be derived using its physical meanings.

IV. Numerical Example

In this example, we illustrate the calculation of the rates of the first four eigenvalues of the plane frame shown in Fig. 2 with respect to the in-span support location parameter b. Figure 2 shows a rect-

Table 1 Rate of eigenvalue with respect to in-span support location

Mode no.	1	2	3	4
Support reaction $\hat{\lambda}$	1537.833	−1591.020	−8786.684	19391.100
Rotation in Z sense $\hat{y}'(b)$	3.32456×10^{-3}	-3.83999×10^{-3}	1.85539×10^{-2}	1.00022×10^{-2}
Rate from Eq. (17) $2\hat{\lambda}\hat{y}'(b)$	10.226	14.985	−326.054	−387.907
ω^2 ($b = 6.00 + 0.01$)	35.010907	262.978052	1689.757133	2257.098567
ω^2 ($b = 6.00$)	34.908397	262.827746	1693.077528	2260.955922
Rate from FFD $[\omega^2(6.01) - \omega^2(6.00)]/0.01$	10.251	15.030	−332.039	−385.736

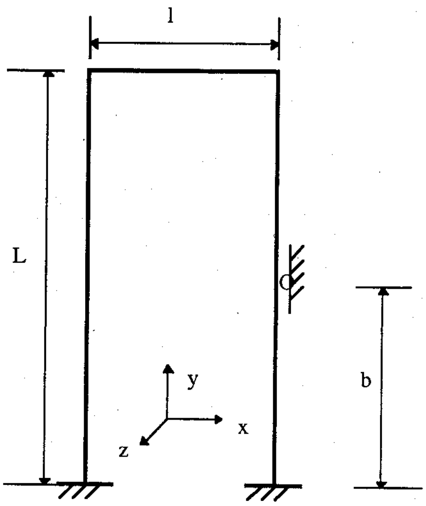

Fig. 2 Plane frame with an in-span support.

angular plane frame with an in-span support. Now let us assume that the plane frame has the physical properties: Young's modulus $= 2.1 \times 10^{11}$ N/m^2, mass density $= 7.8 \times 10^3$ kg/m^3, cross-sectional area $= 0.10 \times 0.10 = 0.01$ m^2, Poisson's rate $= 0.3$, $L = 12$ m, $l = 4$ m, $b = 6.00$ m.

The first four natural frequencies and the corresponding mode shapes are obtained using finite element method (FEM) by dividing the plane frame uniformly into 56 beam elements. Note that the mode shapes are assumed to be normalized to unit modal mass. Since for a beam element via FEM the slope of mode shapes $\hat{y}'(b)$ at a grid point is just the available rotation in Z sense of that grid point, the rate of eigenvalues with respect to b can be evaluated directly using Eqs. (13) and (14). The results are summarized in Table 1.

For the purpose of comparison, the rate of eigenvalues are also computed using forward finite difference (FFD) method with $\Delta b = 0.01$ and the results are included in the fourth column of Table 1. The FFD method used here can be stated as

$$\frac{d\omega_i^2}{db} = \frac{\omega_i^2(b + \delta b) - \omega_i^2(b)}{\delta b} \tag{15}$$

As can be seen in the last column of Table 1, the results obtained using Eqs. (13) and (14) agree well with the results obtained using the FFD method.

V. Concluding Remarks

A new method for deriving the formulas of eigenvalue rate with respect to in-span support location is presented in this Note. Based on the generalized variational principle, the approach presented can be easily extended to include the continuum and other structural members. The eigenvalue rate with respect to in-span support location can be used to find desirable locations of supports to maximize the fundamental natural frequencies of a structure member or can be used for the purpose of reanalysis of the structure with modified support locations.

Acknowledgment

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New Method for the Improvement of Measured Modes Through Orthogonalization

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Introduction

FOR large space structures, the mode shapes obtained from tests seldom have acceptable compatibilities with the analytically derived model. Generally, the orthogonality check of the measured modes by means of the analytical mass matrix results in some off-diagonal elements that cannot be ignored. We present a new method to perform the orthogonalization of the measured modes, assuming that the analytical mass matrix is exactly known. It is noted that the selection of the mass matrix as a reference base in the orthogonalization process is not unique; the problem of satisfying orthogonality has been attacked from different directions: Some methods[1–5] consider that the measured modes are more reliable and use them to adjust the analytical mass matrix to achieve orthogonality. Others[6–8] consider that the analytically derived mass matrix is correct and adjust the measured modes so that the orthogonalization condition is satisfied. After the orthogonalization is achieved, the analytical stiffness matrix can also be improved. Discussions on the degree of applicability of the two distinct approaches in the orthogonalization process can be found in Refs. 9 and 10.

For the orthogonalization of the measured modes in this Note, we seek a solution in an optimal way following the methodology of Baruch and Bar Itzhack.[8] However, we do not impose any restrictions on the type of the transformation matrix, triangular or symmetric, and thus the resulting adjusted modes are closest to the measured ones. Higher confidences to the lower modes can be considered by a diagonal weighting matrix. The adjusted modes are orthogonal to the rigid-body ones, if any. A simple closed-form solution, as well as an iterative procedure, is derived and demonstrated with an example; a comparison with the Targoff/Baruch method[7,8] is also presented.

Method Description

The problem can be mathematically expressed as follows: Given a mass matrix of a structure M $(n \times n)$, a rigid-body modal matrix ϕ_r $(n \times r)$ that is orthogonal with respect to the mass matrix, and a measured modal matrix ϕ_m $(n \times m)$, $m < n$, determine an adjusted flexible modal matrix ϕ $(n \times m)$ that satisfies the orthogonality condition:

$$\phi_r^t M \phi = 0 \tag{1}$$

$$\phi^t M \phi = I \tag{2}$$

where the superscript t denotes matrix transpose, and I represents a unit matrix. Assuming

$$\phi = [\phi_r \phi_m] \begin{bmatrix} R \\ C \end{bmatrix} \tag{3}$$

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